Série 4a Solutions

Question 4a.1 – Rotation of element

Consider the state of stress given in Figure 4a.1.

Determine the normal and shearing stresses after the element shown has been rotated through:

(a) 25° clockwise.

(b) 10° counterclockwise.

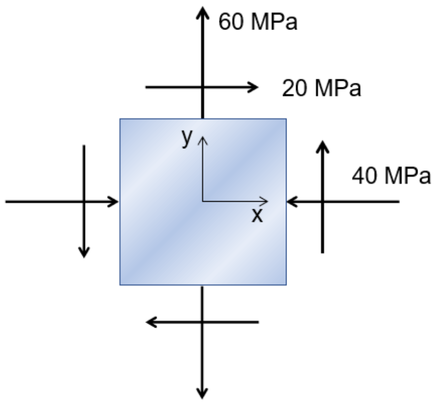


Figure 4a.1 – State of stress on a 2D element

Solution 4a.1

Objectives – what is asked?

Normal and shearing stresses after rotation ($\sigma_{x'}$, $\sigma_{y'}$, $\tau_{x'y'}$)

What is given?

A state of stress, where the normal stresses (σ_x = -40 MPa, σ_y = 60 MPa) and the shearing stress (τ_{xy} = 20 MPa) are known.

The angle of rotation Θ

Principles and formulas

Rotation of stresses and shear:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} cos(2\theta) + \tau_{xy}\, sin(2\theta) \tag{1}$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} cos(2\theta) - \tau_{xy}\, sin(2\theta) \tag{2}$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta) \tag{3}$$

Calculations

(a) For 25° clockwise (-25°):

$$\sigma_{x'} = \frac{-40+60}{2} + \frac{-40-60}{2} cos(2*-25°) + 20\, sin(2*-25°) = -37.5\, MPa \tag{4}$$

$$\sigma_{y'} = \frac{-40+60}{2} - \frac{-40-60}{2} cos(2*-25°) - 20\, sin(2*-25°) = 57.5\, MPa \tag{5}$$

$$\tau_{x\prime y\prime} = -\frac{-40-60}{2} sin(2*-25°) + 20\, cos(2*-25°) = -25.4\, MPa \tag{6}$$

(b) For 10° counterclockwise (10°):

$$\sigma_{x'} = \frac{-40+60}{2} + \frac{-40-60}{2} cos(2*10°) + 20\, sin(2*10°) = -30.1\, MPa \tag{7}$$

$$\sigma_{y'} = \frac{-40+60}{2} - \frac{-40-60}{2} cos(2*10°) - 20\, sin(2*10°) = 50.1\, MPa \tag{8}$$

$$\tau_{x\prime y\prime} = -\frac{-40-60}{2} sin(2*10°) + 20\, cos(2*10°) = 35.9\, MPa \tag{9}$$

Question 4a.2 – Stress and shear in a direction

The grain of a wooden member forms an angle of 15° with the vertical. Consider the state of stress shown in Figure 4a.2.

(a) Determine the in-plane shearing stress parallel to the grain.

(b) Determine the normal stress perpendicular to the grain.

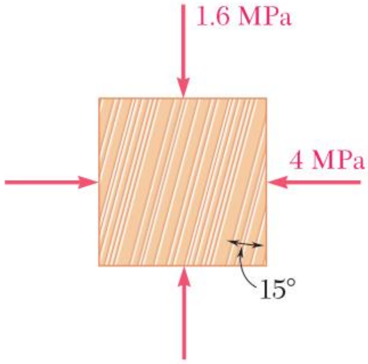


Figure 4a.2 – State of stress on a wooden piece

Solution 4a.2

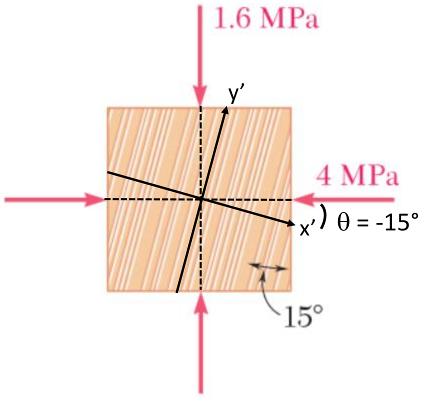
Objectives – what is asked?

Normal and shearing stress in the direction of the grain ($\sigma_{x'}$, $\tau_{x'y'}$)

What is given?

A state of stress, where the normal stresses (σ_x = -4 MPa, σ_y = -1.6 MPa) and the shearing stress (τ_{xy} = 0 MPa) are known.

From the drawing showing the orientation of the grain, the angle of rotation can be derived.



The angle of rotation Θ = -15°

Principles and formulas

Rotation of stresses and shear:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} cos(2\theta) + \tau_{xy}\, sin(2\theta) \quad (1)$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} cos(2\theta) - \tau_{xy}\, sin(2\theta) \quad (2)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta) \quad (3)$$

Calculations

(a) For the shearing stress:

$$\tau_{x\prime y\prime} = -\frac{-4 + 1.6}{2} sin(2 * -15°) - 0 = -0.6\, MPa \quad (4)$$

(b) For the normal stress:

$$\sigma_{x'} = \frac{-4 - 1.6}{2} + \frac{-4 + 1.6}{2} cos(2 * -15°) + 0 = -3.84\, MPa \quad (5)$$

Question 4a.3 – Transformation of 2D stress

For the two-dimensional stress state shown in Figure 4a.3 determine:

(a) The principal stresses.
(b) The principal axes.
(c) The maximum shear stress and the angle for the maximum shear stress.
(d) The stress components exerted on the element obtained by rotating the reference axis counterclockwise 45°.
(e) Strain energy density of the element before and after the 45° rotation (Young's modulus and Poisson's ratio are respectively $E = 200$ GPa **and** $\nu = 0.25$**).**

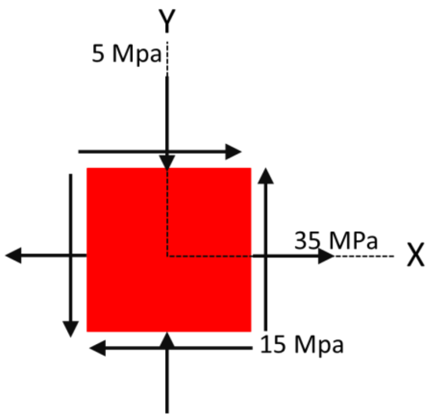


Figure 4a.3 – State of stress on a 2D element

Solution 4a.3

Objectives – what is asked?

(a) The principal stresses
(b) The principal axes
(c) The maximum shear stress and the angle of maximum shear stress
(d) Stresses after rotating 45°
(e) Strain energy density before and after the rotation

What is given?

Stress components:
$\sigma_x = 35\ \text{MPa}$
$\sigma_y = -5\ \text{MPa}$
$\tau_{xy} = 15\ \text{Mpa}$
$E = 200\ \text{GPa}$
$\nu = 0.25$
We assume the material is homogeneous and isotropic.

Principles and formulas

To find the principal stresses and principal axes, we use to fact the they correspond respectively to the eigenvalues and eigenvectors of the stress tensor:

$$\det(\tilde{\sigma} - \lambda \cdot \mathbb{I}) = 0 \tag{1}$$

The formula for the maximum shear stress and rotation of normal and shear stresses will be used as well and the formula for the shear energy density (in 2D):

$$u_0 = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{yx}\right) \tag{2}$$

Calculations and numerical application

(a) Principal Stresses

We calculate the eigenvalues of the stress tensor:

$$\tilde{\sigma} = \begin{pmatrix} 35 & 15 \\ 15 & -5 \end{pmatrix} \text{MPa} \tag{3}$$

For which we solve the following equation:

$$\begin{aligned}\det(\tilde{\sigma} - \lambda \cdot \mathbb{I}) = 0 = \begin{vmatrix} 35-\lambda & 15 \\ 15 & -5-\lambda \end{vmatrix} &= (35-\lambda)(-5-\lambda) - 15^2 \\ &= \lambda^2 - 30\lambda - 400\end{aligned} \tag{4}$$

Which gives us:

$$\lambda = \frac{30 \pm \sqrt{900+1600}}{2} \rightarrow \lambda_1 = 40\ \text{MPa};\ \lambda_2 = -10\ \text{MPa} \tag{5}$$

So the principal stresses are:

$$\sigma_{min} = \sigma_2 = -10\ \text{MPa} \tag{6}$$

$$\sigma_{max} = \sigma_1 = 40\ \text{MPa} \tag{7}$$

Another way of calculating the principal stresses is using the formula provided in the notes:

$$\sigma_{max,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = 15 \pm \sqrt{(20)^2 + (15)^2} \\ = 15 \pm 25 \text{ MPa} \tag{8}$$

(b) Principal axes

We calculate the Eigenvectors of the stress tensor given by Eq. (3). applying the following:

$$\tilde{\sigma} \cdot \vec{v}_{\lambda_i} = \lambda_i \cdot \vec{v}_{\lambda_i} \tag{9}$$

where $\vec{v}_{\lambda_i}$ is the Eigenvector with Eigenvalue λ_i. If we take $\lambda_1 = 40$ MPa:

$$\tilde{\sigma} \cdot \vec{v}_\lambda = \begin{pmatrix} 35 & 15 \\ 15 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 40 \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{cases} 35a + 15b = 40a \\ 15a - 5b = 40b \end{cases} \rightarrow \begin{cases} -5a = -15b \\ 15a = 45b \end{cases} \rightarrow a = 3b \tag{10}$$

which means that the Eigenvector with Eigenvalue $\lambda_1 = 40$ MPa is:

$$\vec{v}_{\lambda = 40 \text{ MPa}} = \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{11}$$

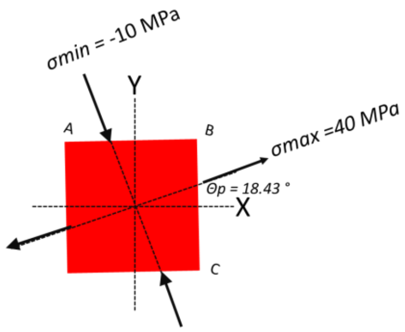
To calculate the other Eigenvector we can redo the calculations in Eq. (10) with $\lambda = -10$ MPa, or we use the fact that both Eigenvectors are orthogonal to each other:

$$\vec{v}_{\lambda = -10 \text{ MPa}} = \vec{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{12}$$

These two Eigenvectors define the direction of the two principal axes. The angle between the horizontal X axis and the principal axis is:

$$\theta_p = \arctan\left(\frac{1}{3}\right) = 18.435° \tag{13}$$

which is graphically shown in the Figure below.



Another way of calculating the direction of the principal axis is to use the formulas derived from class:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(15)}{35 - (-5)} = 0.75 \tag{14}$$

$$2\theta_p = \arctan\left(\frac{3}{4}\right) = 36.86° \rightarrow \theta_p = 18.43° \tag{15}$$

Indeed, we can check that the normal stress exerted on face BC of the Figure above is given by Eq.(16):

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\, sin2\theta \;\rightarrow\; \sigma_{x'} \\ = 15 + 20\cos(36.86°) + 15\sin(36.86°) \tag{16}$$

$$\sigma_{x'} = 40 \text{ MPa} = \sigma_{max} = \sigma_1 \tag{17}$$

(c) The maximum shear stress and the angle for the maximum shear stress

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \sqrt{(20)^2 + (15)^2} = 25 \text{ MPa} \tag{18}$$

$$2\theta_s = 2\theta_p \pm 90° \;\rightarrow \theta_s = 18.43° \pm 45° = \begin{cases} 63.43° \\ -26.57° \end{cases} \tag{19}$$

(d) The stresses after a 45° rotation

We can use the equations for the rotation of the stresses on an element to calculate:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \rightarrow \;\sigma_{x'} = 15 + 20\cos(90°) + 15\sin(90°) \\ = 30 \text{ MPa} \tag{20}$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\;\sin 2\theta \;\rightarrow\; \sigma_{y'} = 15 - 20\cos(90°) - 15\sin(90°) \\ = 0 \text{ MPa} \tag{21}$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \rightarrow \;\tau_{x'y'} = -20\sin(90°) + 15\cos(90°) \\ = -20 \text{ MPa} \tag{22}$$

(e) The strain energy density after and before 45 ° rotation

$$u_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{yx}\right) = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2\right) - \frac{\nu}{E}\sigma_x\sigma_y + \frac{\tau_{xy}^2}{2G} \tag{23}$$

Where, E is the Young's modulus, $G = \frac{E}{2(1+\nu)}$ is the shear modulus and ν is the Poisson's ratio of the material, σ_x, σ_y are respectively the normal stresses parallel to the x and y-axis, of the material and τ_{xy} is the shear stress along the (xy) plane

$$u_0 = 4750 \;{}^{\text{J}}\!/_{\text{m}^3} \tag{24}$$

The strain energy should not change after a rotation. This is understandable because changing the axis does not modify the stress state. Therefore, we can choose any set of axis we want to calculate the energy, e.g. the principal axes, for which Eq. (23) is:

$$u_0 = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2\right) - \frac{\nu}{E}\sigma_1\sigma_2 = 4750 \;{}^{\text{J}}\!/_{\text{m}^3} \tag{25}$$

Question 4a.4 – Stress transformation in 3D

For the stress state shown in Figure 4a.4 with the stress tensor of $\begin{pmatrix} 22 & 6 & 10 \\ 6 & 13 & 5 \\ 10 & 5 & 5 \end{pmatrix}$ MPa determine

(a) The three principal stresses and principal axes.

(b) The maximum shear stress.

(c) Calculate the Von Mises stress.

(d) If the yield strength $\sigma_{yield} = 250$ MPa, calculate the safety factor (SF_{VM}) using Von Mises criterion.

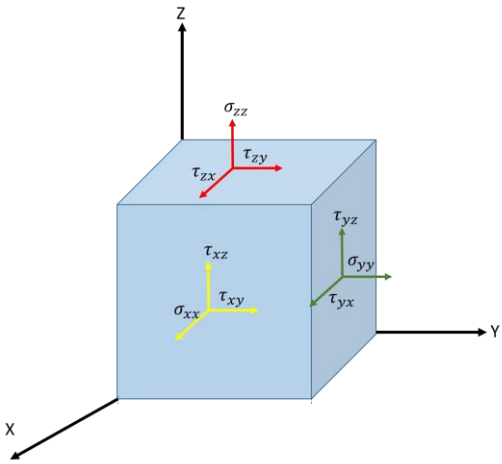


Figure 4a.4 – Stress state on a 3D element

Solution 4a.4

Objectives – what is asked?

(a) The three principal axes and principal stresses.
(b) The maximum shear stress.
(c) The Von Mises stress.
(d) The factor of safety.

What is given?

The stress tensor $\tilde{\sigma} = \begin{pmatrix} 22 & 6 & 10 \\ 6 & 13 & 5 \\ 10 & 5 & 5 \end{pmatrix}$ MPa
Tensile yield strength $\sigma_{yield} = 250$ MPa
The material is homogeneous and isotropic

Principles and formulas

The principal axes and principal stresses are derived from the stress tensor (eigenvalues, eigenvectors). The maximum shear stress in 3D is given by:

$$\tau_{max} = \max\left(\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_1 - \sigma_3}{2}, \frac{\sigma_2 - \sigma_3}{2}\right) \tag{1}$$

The Von Mises stress in 3D and the corresponding factor of safety:

$$\sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$
$$SF_{VM} = \frac{\sigma_{\text{yield}}}{\sigma_{VM}} \tag{2}$$

Calculations and numerical application

(a) Principal Stresses

The principal Stresses are obtained by finding the Eigenvalues of the given stress tensor using:

$$\det(\tilde{\sigma} - \lambda\mathbb{I}) = 0 = \begin{pmatrix} 22-\lambda & 6 & 10 \\ 6 & 13-\lambda & 5 \\ 10 & 5 & 5-\lambda \end{pmatrix} \tag{3}$$

After expanding the determinant we find the following equation:

$$\lambda^3 - 40\lambda^2 + 300\lambda = 0 \tag{4}$$

One solution is $\lambda = 0$ and the other two come from the following:

$$\lambda^2 - 40\lambda + 300 = 0 \rightarrow \lambda = \frac{40 \pm \sqrt{1600 - 1200}}{2} = \begin{cases} 30 \text{ MPa} \\ 10 \text{ MPa} \end{cases} \tag{5}$$

So the principal stresses are:

$$\sigma_1 = 30 \text{ MPa};\ \sigma_2 = 10 \text{ MPa};\ \sigma_3 = 0 \text{ MPa} \tag{6}$$

To find the principal axes we use

$$\begin{pmatrix} 22 & 6 & 10 \\ 6 & 13 & 5 \\ 10 & 5 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 30\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 22a + 6b + 10c \\ 6a + 13b + 5c \\ 10a + 5b + 5c \end{pmatrix} \rightarrow \begin{cases} 8a = 6b + 10c \\ 6a = 17b - 5c \\ 10a = -5b + 25c \end{cases} \tag{7}$$

On solving this system of equations, we get:

$$a = 2c \;\; \& \;\; b = c \tag{8}$$

The first principal plane lies along the direction of the vector

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \tag{9}$$

We do the same with the second principal stress:

$$\begin{pmatrix} 22 & 6 & 10 \\ 6 & 13 & 5 \\ 10 & 5 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 10 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 22a + 6b + 10c \\ 6a + 13b + 5c \\ 10a + 5b + 5c \end{pmatrix} \rightarrow \begin{cases} -12a = 6b + 10c \\ 6a = -3b - 5c \\ 10a = -5b + 5c \end{cases} \rightarrow \vec{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \tag{10}$$

And the last one we can do the same thing or calculate the only vector family that is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} \tag{11}$$

(b) Maximum shear stress

The maximum shear stresses for each plane is:

$$\tau_{\max\,(1,2)} = \frac{\sigma_1 - \sigma_2}{2} = \frac{30 - 10}{2} = 10 \text{ MPa} \tag{12}$$

$$\tau_{\max\,(2,3)} = \frac{\sigma_2 - \sigma_3}{2} = \frac{10 - 0}{2} = 5 \text{ MPa} \tag{13}$$

$$\tau_{\max\,(1,3)} = \frac{\sigma_1 - \sigma_3}{2} = \frac{30 - 0}{2} = 15 \text{ MPa} \tag{14}$$

Overall maximum shear stress is $\tau_{\max\,overall} = 15$ MPa.

(c) Von Mises stress is calculated using the formula

$$\sigma_{VM} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \tag{15}$$

Substituting values of σ_1, σ_2 *and* σ_3 in this equation. We get

$$\sigma_{VM} = \frac{1}{\sqrt{2}} \sqrt{(30 - 10)^2 + (10 - 0)^2 + (0 - 30)^2} = 10\sqrt{7} \approx 26.5 \text{ MPa} \tag{16}$$

(d) Safety Factor is given by

$$SF_{VM} = \frac{\sigma_{\text{yield}}}{\sigma_{VM}} \approx \frac{250}{26.5} \approx 9.45 \tag{17}$$

Question 4a.5 – Principal stresses and maximum shear directions

Consider the state of stress on an element as given below (shown in Figure 4a.5).

(a) Determine the principal axes (calculate the angle between the x-axis shown in Figure 4a.5 and the principal axes).
(b) Determine the principal stresses and draw them on a rotated element.
(c) Determine the orientation of the axes of minimum/maximum in-plane shearing stress (calculate the angle between the x-axis shown in Figure 4a.5 and the axes of minimum and maximum shear stress).
(d) Determine the value of the maximum in-plane shearing stress.
(e) Determine the corresponding normal stresses.

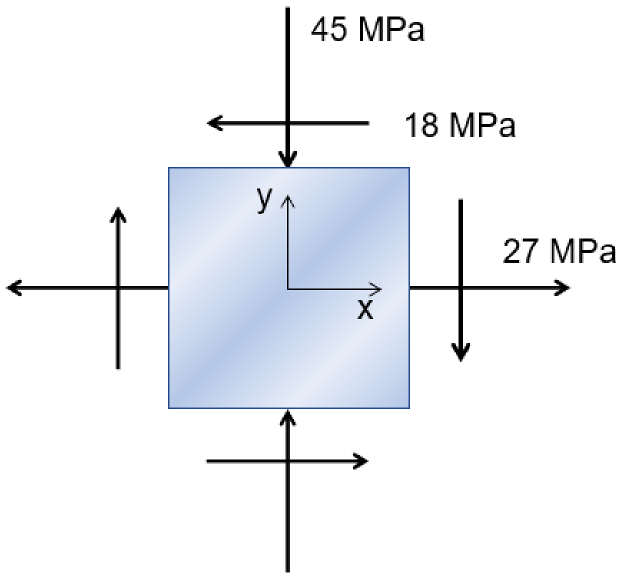


Figure 4a.5 – Stress state on an element

Solution 4a.5

Objectives – what is asked?

(a) The orientation of the principal planes
(b) The principal stresses $\sigma_{max,min}$
(c) The orientation of the planes of maximum in-plane shearing
(d) The value of the maximum in-plane shearing stress, τ_{max}
(e) The normal stress in the orientation of maximum shear stress

What is given?

(a) - (e) A state of stress, where the normal stresses in x (σ_x = 27 MPa, σ_y = -45 MPa) and the shearing stress (τ_{xy} = -18 MPa) are known.

Principles and formulas

Angle between the principal axes and the current axes:

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{1}$$

Principal stresses:

$$\sigma_{max,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{2}$$

The maximum shear stress:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{3}$$

Calculations and numerical applications

(a) The angle between the original X-axis and the principal axes can be calculated in a straightforward manner with the formula given above.

$$2\theta_p = \arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = \arctan\left(\frac{-2 * 18}{27 - (-45)}\right) = -26.57° \tag{4}$$

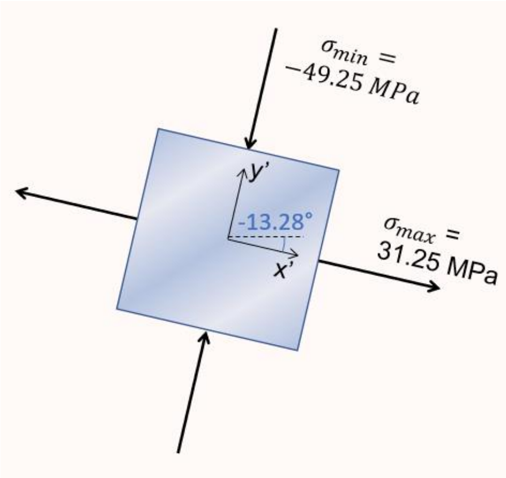
Therefore, the angle between the original X-axis and the first principal axis is -13.28°. The other principal axis is orthogonal and therefore at 90° from the first principal axis (at 76.72° from the original X-axis).

(b) The principal stresses as well:

$$\sigma_{max,min} = \frac{27 - 45}{2} \pm \sqrt{\left(\frac{27 - (-45)}{2}\right)^2 + (-18)^2} = -9 \pm 40.25\ MPa \tag{5}$$

$$\begin{aligned} \sigma_{max} &= 31.25\ MPa \\ \sigma_{min} &= -49.25\ MPa \end{aligned} \tag{6}$$

Drawing them on a rotated element:



Note: it is also possible to use the method seen in exercise 5.3 (i.e. calculating the Eigenvalues and Eigenvectors of the stress tensor). We use it hereafter to validate our results. The stress tensor is defined here as follows:

$$\tilde{\sigma} = \begin{pmatrix} 27 & -18 \\ -18 & -45 \end{pmatrix} \text{ MPa} \tag{7}$$

For which we solve the following equation:

$$\det(\tilde{\sigma} - \lambda \cdot \mathbb{I}) = 0 = \begin{vmatrix} 27-\lambda & -18 \\ -18 & -45-\lambda \end{vmatrix} = \\ (27-\lambda)(-45-\lambda) - 18^2 = \lambda^2 + 18\lambda - 1539 = 0 \tag{8}$$

$$\lambda = \frac{-18 \pm \sqrt{18^2 + 4 * 1 * 1539}}{2} \rightarrow \lambda_1 = 31.25 \text{ MPa};\ \lambda_2 = -49.25 \text{ MPa} \tag{9}$$

The principal stresses are the Eigenvalues from the stress tensor:

$$\sigma_{min} = \sigma_2 = -49.25 \text{ MPa} \\ \sigma_{max} = \sigma_1 = 31.25 \text{ MPa} \tag{10}$$

And the directions of the principal axes correspond to the Eigenvectors of the stress tensor, which are defined as follows.

$$\tilde{\sigma} \cdot \vec{v}_{\lambda_i} = \lambda_i \cdot \vec{v}_{\lambda_i} \tag{11}$$

with $\vec{v}_{\lambda_i}$ being the Eigenvector with Eigenvalue λ_i. For $\lambda_1 = 40$ MPa:

$$\tilde{\sigma} \cdot \vec{v}_\lambda = \begin{pmatrix} 27 & -18 \\ -18 & -45 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 31.25\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{cases} -18b = 4.25a \\ -18a = 76.25b \end{cases} \rightarrow a = -4.236b \tag{12}$$

$$\vec{v}_1 = \begin{pmatrix} -4.236 \\ 1 \end{pmatrix} \tag{13}$$

The angle between the x axis and the first principal axis is:

$$\theta_p = \arctan\left(\frac{1}{-4.236}\right) = -13.28° \tag{14}$$

(c) We know that the axes of minimum and maximum shearing stress are found at 45° from the principal axes.

$$\theta_s = \theta_p \pm 45° = -13.28° \pm 45° = \begin{cases} 31.72° \\ -58.28° \end{cases} \tag{15}$$

(d) The maximum shearing stress:

$$\tau_{max} = \sqrt{\left(\frac{27-(-45)}{2}\right)^2 + (-18)^2} = 40.25\ MPa \tag{16}$$

(e) In the direction of maximum (and minimum) shear stress, the normal stresses are equal and have for value the average value of the normal stresses (which is constant no matter the direction)

$$\sigma_{\tau max} = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = -9\ MPa \tag{17}$$

Question 4a.6 – Safety factor and maximal load

Consider the same state of stress as in the previous exercise (4a.5), except that the shear stress on the element (τ_{xy}) is unknown (see Figure 4a.6). The other components of the stress (σ_x and σ_y) remain as previously described (σ_x = 27 MPa, σ_y = -45 MPa). The material to which this element belongs has a yield stress of σ_{yield} = 150 MPa.

Using the Von Mises criterion, determine the maximal value for τ_{xy} for which the safety factor is still equal or above to 2.

Hint: consider that the principal stresses can be written in a simplified manner as:

$$\sigma_{1,2} = \sigma_{ave} \pm R$$

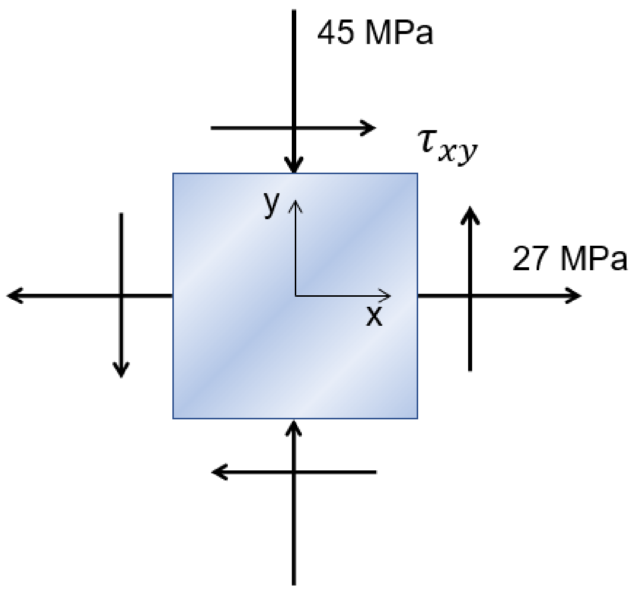


Figure 4a.6 – Stress state on an element with unknown τ_{xy}

Solution 4a.6

Objectives – what is asked?

The maximal value for τ_{xy} with a safety factor superior or equal to 2.

What is given?

A state of stress, where the normal stresses in x ($\sigma_x = 27$ MPa, $\sigma_y = -45$ MPa)

Principles and formulas

The Von Mises stress (in 2D):

$$\sigma_{VM} = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} \tag{1}$$

The safety factor:

$$SF_{VM} = \frac{\sigma_{yield}}{\sigma_{VM}} \tag{2}$$

Calculations

Consider the definition of the Von Mises stress in 2D and the safety factor:

$$\sigma_{VM} = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} = \frac{\sigma_{yield}}{SF_{VM}} \tag{3}$$

To simplify, we consider the following relations:

$$\begin{gathered} \frac{\sigma_x + \sigma_y}{2} = \sigma_{ave} = constant \\ \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 = R^2 = constant \end{gathered} \tag{4}$$

These stem from the fact that it can be shown that all planar stresses are found on a circle of radius R with center σ_{ave}. Using these relations to simplify the equation of the principal stresses we write:

$$\sigma_{1,2} = \sigma_{ave} \pm R \tag{5}$$

The Von Mises stress can then be simplified:

$$\sigma_{VM}^2 = (\sigma_{ave} + R)^2 - (\sigma_{ave} + R)(\sigma_{ave} - R) + (\sigma_{ave} - R)^2 \tag{6}$$

$$\sigma_{VM}^2 = \sigma_{ave}^2 + 2\sigma_{ave}R + R^2 - \sigma_{ave}^2 + R^2 + \sigma_{ave}^2 - 2\sigma_{ave}R + R^2 \tag{7}$$

$$\sigma_{VM}^2 = \sigma_{ave}^2 + 3R^2 = \frac{\sigma_{yield}^2}{SF_{VM}^2} \tag{8}$$

R is the only term that contains τ_{xy}, so we isolate it.

$$R^2 = \frac{1}{3}\left(\frac{\sigma_{yield}^2}{SF_{VM}^2} - \sigma_{ave}^2\right) \tag{9}$$

Expressing R and σ_{ave} back as a function of the stresses and solving for τ_{xy} gives:

$$\tau_{xy} = \sqrt{\frac{1}{3}\left(\frac{\sigma_{yield}^2}{SF_{VM}^2} - \left(\frac{\sigma_x + \sigma_y}{2}\right)^2\right) - \left(\frac{\sigma_x - \sigma_y}{2}\right)^2} \quad (10)$$

Numerical application

With numerical values, the maximal shear stress to have a safety factor of 2 is:

$$\tau_{xy} = \sqrt{\frac{1}{3}\left(\frac{150^2}{4} - \left(\frac{27-45}{2}\right)^2\right) - \left(\frac{27+45}{2}\right)^2} = 23.49\ MPa \quad (11)$$

